JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 16, No. 4, July–August 1993

Singular Control in Minimum Time Spacecraft Reorientation

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Time-optimal solutions for the reorientation of an inertially symmetric rigid spacecraft with independent three-axes controls are investigated. All possible optimal control strategies are identified. These include bang-bang solutions, finite-order singular arcs and infinite-order singular arcs. Necessary conditions for optimality of finite-order singular arcs are presented. Nonoptimality of principal-axis rotation for rest-to-rest minimum time reorientation is proved. Numerical examples of time-optimal solutions with finite- and infinite-order singular arcs are presented. A simple example of "a car in a parking lot" is discussed. It clearly illustrates that infinite-order singular control may be the "unique" solution to an optimal control problem and cannot be ignored when choosing the optimal control strategies.

I. Introduction

OPTIMAL rigid body rotational maneuvers have been studied in detail by many researchers in the context of spacecraft attitude control systems.[1-5] Minimum time or minimum control effort rotational maneuvers have been the main objectives in these studies.

Recently, the minimum time reorientation problem has been studied by Li and Bainum,[5] the approach being to solve a sequence of minimum control effort problems with fixed final times. While subsequently reducing the final time t_f, it is assumed that the smallest value of t_f for which a solution could be obtained is close to the actual minimum final time and that the obtained control logic is "close" to the optimal minimum time solution. Numerical results suggested that, for the specific spacecraft model considered in this paper, principal axis rotation is time optimal at least for some rest-to-rest reorientation problems.

It is clear that this result is based on some continuity assumption in the solution with respect to the prescribed final time, a dangerous assumption especially as the minimum time trajectory refers to an abnormal solution point of the minimum control effort problem. But even if this continuity assumption is satisfied, the minimum final time obtained by the sequential numerical solution of minimum control effort problems may be just a local minimum final time depending upon the specific choice of cost function.

In a later study by Bilimoria and Wie,[1] a variety of minimum time rest-to-rest reorientation problems has been solved and the nonoptimality of principal axis rotation has been shown via numerical solutions of associated two-point boundary value problems (TPBVPs). However, the possibility of singular control has not been analyzed, except for the rather trivial case where all controls are singular simultaneously.

In this paper the authors investigate the problem of reorienting an inertially symmetric spacecraft in minimum time with all angular positions and velocities given at initial time and some or all angular positions and velocities given at final time. Three bounded independent control torques are assumed, with the control axes aligned along a prescribed set of principal axes.

Emphasis is laid on identifying all possible optimal control logics for the previously stated problem. This includes bang-bang control arcs, finite-order singular arcs, and infinite-order singular arcs.

Received Oct. 23, 1991; revision received Oct. 12, 1992; accepted for publication Oct. 16, 1992.

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Higher order necessary conditions for optimality of finite-order singular arcs are examined, using Goh transformations of the associated Accessory Minimum Problem. Numerical examples of optimal solutions with finite- and infinite-order singular arcs are presented. Examples with bang-bang structure can be found in Ref. 1.

For rest-to-rest maneuvers, principal-axis rotations are identified as finite-order singular arcs, the nonoptimality of which is shown in Sec. VI.B. It is shown that principal axis rotations can appear in the synthesis of time optimal solutions only as realizations of infinite-order singular control arcs. In this case, the control history that furnishes minimum final time is not determined uniquely.

II. Dynamical System

Consider an inertially symmetric rigid spacecraft with the control axes aligned with a prescribed set of principal axes. The Euler rotational equations of motion can easily be derived as

$$\frac{d\omega}{dt} = u \tag{1}$$

where

$$\omega^T = [\omega_1, \omega_2, \omega_3]$$

is the vector of angular velocities of the rigid body about the three body principal axes (shown in Fig. 1) represented in the body coordinate frame and t is the clock-time. The three independent components of control vector

$$u^T = [u_1, u_2, u_3]$$

are confined to the fixed limits

$$-u_{i,\max} \le u_i \le u_{i,\max} \tag{2}$$

In physical terms $u_i = (T_i/I)$, where T_i, $i = 1, 2, 3$ are the three independent control torques represented in the body coordinate frame and I denotes the moments of inertia of the spacecraft about the chosen principal axes. The kinematic equations of motion can be written as

$$\frac{dq}{dt} = \frac{1}{2}\Omega q \tag{3}$$

where

$$\Omega = \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix} \tag{4}$$

Here, the quaternion vector

$$q^T = [q_0, q_1, q_2, q_3]$$

is used instead of the Euler angles to prevent singularities and to simplify the kinematic differential equations. However, a drawback in the use of quaternions is the increase in the state vector dimension by 1.

The physical meaning of the quaternions is as follows:

$$q_0 = \cos\Phi/2$$

$$q_i = \cos\epsilon_i \sin\Phi/2; \qquad i = 1,2,3$$

where $(\epsilon_1, \epsilon_2, \epsilon_3)$ are the angles between the Euler axis and an inertially fixed orthogonal coordinate system (without loss of generality coincident with the body axis at initial time), as shown in Fig. 1. It may be noted that these are also the angles between the body axes and the Euler axis. Note that a physical meaning of the quaternion vector exists only for the case where $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$. At the starting point of the trajectory, this constraint will automatically be satisfied if only the initial values $q(t_0)$ of the quaternion vector are chosen appropriately. Throughout the trajectory, the skew-symmetry of matrix Ω defined in Eq. (4) ensures that $d/dt(q^Tq) = 0$, so that the Euclidean norm of the quaternion vector is preserved.

A principal axis rotation is defined as a rotation during which the Euler axis is always aligned with a prescribed body principal axis (also the same as a control axis).

In the course of this paper, we will find it very helpful to introduce the following matrices:

$$D_1 := \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{5}$$

$$D_2 := \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{6}$$

$$D_3 := \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{7}$$

The benefit of these definitions lies in the fact that $\Omega = \omega_1 D_1 + \omega_2 D_2 + \omega_3 D_3$, along with the simple relations:

$$\begin{aligned} D_i^T &= -D_i \\ D_i D_i &= -I \\ D_1 D_2 &= -D_3 \\ D_2 D_3 &= -D_1 \\ D_3 D_1 &= -D_2 \\ D_i D_j &= -D_j D_i, \quad i \neq j \end{aligned} \tag{8}$$

where $i, j \in \{1, 2, 3\}$.

III. Optimal Control Problem

Let us now consider the following optimal control problem [minimum time spacecraft reorientation (MTSR)]:

$$\min_{u \in (PWC[t_0,t_f])^3} t_f \tag{9}$$

subject to the dynamical system of Eqs. (1) and (3), the initial conditions

$$\begin{aligned} \omega(0) &= \omega^0, \qquad \omega^0 \in \boldsymbol{R}^3 \quad \text{given} \\ q(0) &= q^0, \qquad q^0 \in \boldsymbol{R}^4 \quad \text{given} \end{aligned} \tag{10}$$

the final conditions

$$\Psi\left[\omega(t_f), q(t_f)\right] = 0 \tag{11}$$

and the control constraints of Eq. (2) with $u_{i,\max} > 0$, $i = 1, 2, 3$.

In Eq. (9), $PWC[t_0, t_f]$ denotes the set of all piecewise continuous functions on the interval $[t_0, t_f]$. The analysis in the remainder of this paper remains exactly the same if we replace $PWC[t_0, t_f]$ by $L_2[t_0, t_f]$, the Hilbert space of all quadratically integrable functions on the interval $[t_0, t_f]$, as the set of admissible control functions.

From Eq. (3) and the antisymmetry of Ω given in Eq. (4), it follows readily that $d/dt(q^Tq) = 0$. Hence, the dynamical system of Eqs. (1) and (3) of problem MTSR is obviously not controllable. (On the subject of controllability of nonlinear systems, the reader is referred to Refs. 6 and 7.) For this reason the function

$$\Psi: \begin{cases} \boldsymbol{R}^3 \times \boldsymbol{R}^4 & \rightarrow \boldsymbol{R}^k \\ \left[\omega(t_f), q(t_f)\right] & \mapsto \Psi\left[\omega(t_f), q(t_f)\right] \end{cases} \tag{12}$$

representing the final conditions of Eq. (11) has to satisfy

$$k \leq 6 \quad \text{and} \quad \text{rank}\left(\frac{\partial \Psi}{\partial q(t_f)}\right) \leq 3 \tag{13}$$

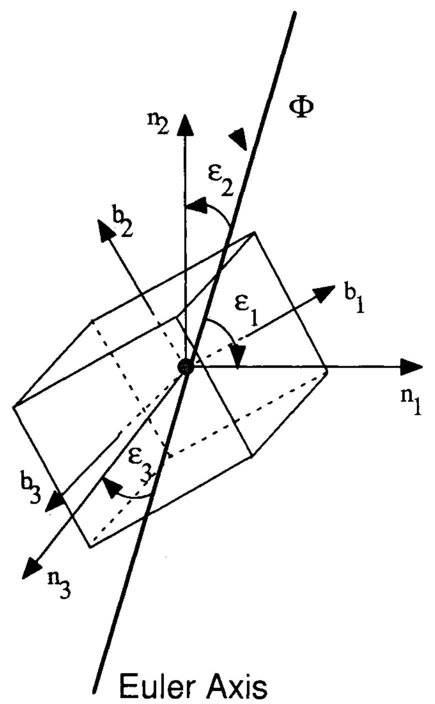

Fig. 1 Definition of coordinate axes and Euler parameters.

This represents a necessary condition for the existence of a nondegenerate solution of problem MTSR. It can be seen from the costate equations, stated later in Eq. (16), that the lack of controllability in the q equations translates into the same lack of controllability in the λ_q equations. This is an important point: to set up a consistent boundary value problem for generating numerical solutions. It is also interesting to note that this lack of controllability is not inherent to the underlying physical problem but is introduced only by the representation of rotation angles in terms of Euler parameters.

IV. Optimality Conditions

Let the Hamiltonian H be defined by

$$H = \lambda_\omega^T u + \tfrac{1}{2}\lambda_q^T \Omega_q \tag{14}$$

Then the costate equations $\dot{\lambda}_x = -(\partial H/\partial x)^T$, $x \in \{\omega, q\}$ take the explicit form

$$\dot{\lambda}_\omega = -\frac{1}{2}\begin{bmatrix} q^T D_1^T \lambda_q \\ q^T D_2^T \lambda_q \\ q^T D_3^T \lambda_q \end{bmatrix} \tag{15}$$

$$\dot{\lambda}_q = -\tfrac{1}{2}\Omega^T \lambda_q \tag{16}$$

As all three controls u_1, u_2, u_3 appear only linearly in the Hamiltonian Eq. (14), the Pontryagin minimum principle $u = \arg\min H$ (see Refs. 8-11) yields for $i = 1, 2, 3$:

$$u_i = \begin{cases} +1 & \text{if } \lambda_{\omega_i} < 0 \\ -1 & \text{if } \lambda_{\omega_i} > 0 \\ \text{singular} & \text{if } \lambda_{\omega_i} \equiv 0 \end{cases} \tag{17}$$

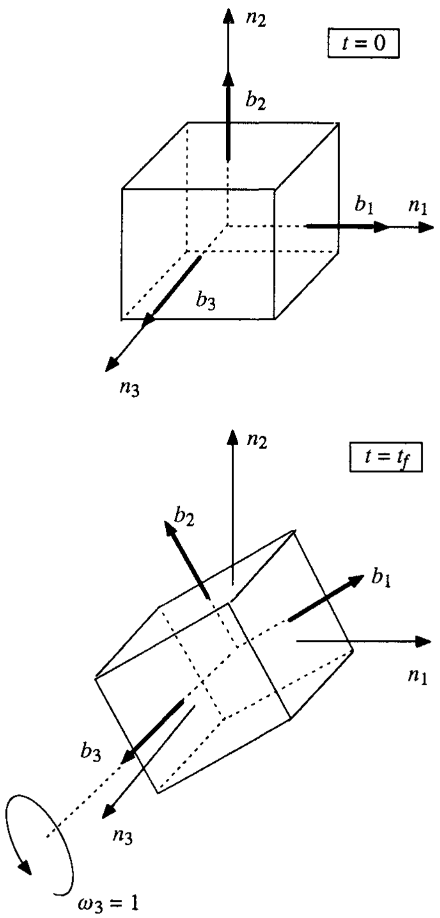

Fig. 2 Nontrivial reorientation problem that leads to infinite-order singular control.

At times t_s where any of the control variables u_i switch between the possible control logics of Eq. (17), a necessary condition for optimality is given by the continuity of the Hamiltonian

$$H|_{t_s^+} - H|_{t_s^-} = 0 \tag{18}$$

In practice, this condition is used to determine the location of switching times. The transversality conditions associated with the boundary conditions of Eq. (11) are given by

$$\lambda_\omega^T(t_f) = \nu^T \frac{\partial \Psi}{\partial \omega(t_f)} \tag{19}$$

$$\lambda_q^T(t_f) = \nu^T \frac{\partial \Psi}{\partial_q(t_f)} \tag{20}$$

where Ψ is given by Eq. (12) and $\nu \in \mathbf{R}^k$ is a constant multiplier vector. The optimal final time t_f is determined by

$$H|_{t_f} = -1 \tag{21}$$

V. Transformation of the Costate Dynamics

For analyzing the singular control cases $\lambda_{\omega_i} \equiv 0$ and for a better understanding of the costate dynamics, we will find the following transformation of the costate dynamics helpful. Let $S = [S_1, S_2, S_3]^T$ be the vector of switching functions, i.e., for $i = 1, 2, 3$:

$$S_i := \lambda_{\omega_i} \tag{22}$$

Then, by Eq. (15), the first time derivative of S_i is given by

$$\dot{S}_i = d_i$$
$$d_i := -\tfrac{1}{2} q^T D_i^T \lambda_q \tag{23}$$

Using Eqs. (1), (3), and (16), along with the simple relations given in Eq. (8), it is easy to verify that $d^T := [d_1, d_2, d_3]$ satisfies the differential equation

$$\dot{d} = \hat{\Omega} d \tag{24}$$

with

$$\hat{\Omega} = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} \tag{25}$$

This means that the dynamics of d can be expressed completely in terms of ω and d itself. Vector d also carries all of the information necessary to determine the optimal control. Equation (15) can be rewritten as

$$\dot{\lambda}_\omega = d \tag{26}$$

This implies that the state/costate system of Eqs. (1), (3), (15), and (16) can be substituted equivalently by the simpler system of Eqs. (1), (3), (26), and (24). In analyzing the singular control cases, we will make extensive use of this transformation.

VI. Singular Control

Control u_i is called singular on some nonzero time interval $[\tau_1, \tau_2]$ if the associated switching function S_i is identically zero on $[\tau_1, \tau_2]$. As S_i is independent of controls, successive differentiation of this identity with respect to time is well-defined and can be used to obtain additional conditions. We have

$$S = \lambda_\omega \tag{27}$$

$$\dot{S} = d \tag{28}$$

$$\ddot{S} = \hat{\Omega} d \tag{29}$$

$$S^{(3)} = (\hat{\Omega}^2 + \dot{\hat{\Omega}}) d \tag{30}$$

with $\hat{\Omega}$ given by Eq. (25). Conditions on singular optimal control are obtained from subsequently setting $S=0$, $\dot{S}=0, \ldots$ and so on.

It is clear that all higher derivatives of S (if they exist) are linear in d. In particular, this implies that all derivatives of S are automatically zero if only $d=0$.

Before we proceed with analyzing the singular control cases for problem MTSR, it should be noted that not all possible combinations of singular and nonsingular controls need to be considered. Physically, it is clear that problem MTSR remains essentially unchanged if the coordinate axes are interchanged. This implies, for example, that, in the case where only one control is singular, it can be assumed without loss of generality that control u_1 is singular, whereas u_2, u_3 are nonsingular.

In the following three sections a complete analysis of the singular control cases with one, two, and all three controls being singular is presented.

A. One Control Singular

Without loss of generality we can assume that control u_1 is singular and controls u_2, u_3 are nonsingular. Two possible cases have to be distinguished, namely, $d_1^2+d_2^2+d_3^2=0$ and $d_1^2+d_2^2+d_3^2\neq 0$.

If $d_1^2+d_2^2+d_3^2=0$, then we obtain singular control of infinite order for u_1. Explicitly, we get

$$\lambda_{\omega_1}=0$$

$$d_1=0$$

$$d_2=0$$

$$d_3=0 \tag{31}$$

If conditions of Eq. (31) are satisfied at a single point along the trajectory, then they are satisfied identically throughout the trajectory, irrespective of control u_1. In fact, at each instant of time the numerical value of control u_1 is arbitrary as long as it satisfies the control constraint of Eq. (2) and as long as it steers the state vector to a final position that satisfies the boundary condition of Eq. (11).

If $d_1^2+d_2^2+d_3^2\neq 0$, then we obtain singular control of second order for control u_1. Explicitly, we get

$$\lambda_{\omega_1}=0$$

$$d_1=0$$

$$\omega_3 d_2-\omega_2 d_3=0$$

$$(\omega_1\omega_2+u_3)d_2+(\omega_1\omega_3-u_2)d_3=0 \tag{32}$$

at the beginning of the singular arc and

$$u_1\equiv -2\omega_1\frac{u_2d_2+u_3d_3}{\omega_2d_2+\omega_3d_3} \tag{33}$$

in the interior of the singular arc. Goh's necessary condition for singular control is satisfied if and only if $\omega_2d_2+\omega_3d_3>0$.

In the remainder of this section we will give a short proof of the claims made previously. From Eqs. (27), (28), and (29) we find that $S_1=0$, $\dot{S}_1=0$, and $\ddot{S}_1=0$, implying

$$\lambda_{\omega_1}=0 \tag{34}$$

$$d_1=0 \tag{35}$$

$$\omega_3d_2-\omega_2d_3=0 \tag{36}$$

Using Eq. (35) in Eq. (30), $S_1^{(3)}=0$ yields

$$(\omega_1\omega_2+u_3)d_2+(\omega_1\omega_3-u_2)d_3=0 \tag{37}$$

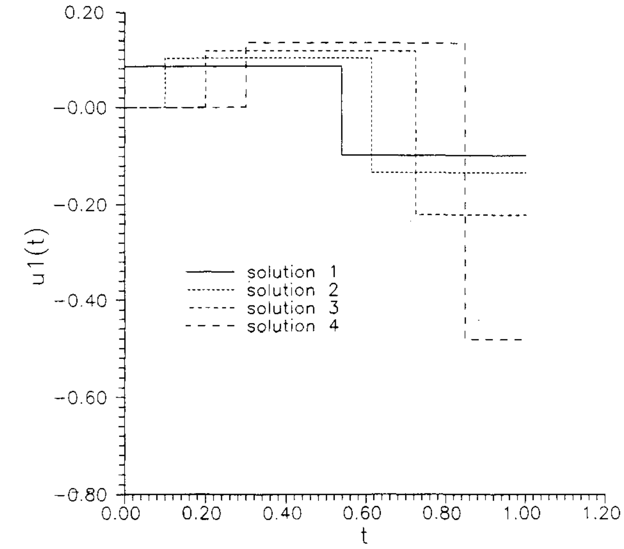

Fig. 3 Time histories of controls u_1 for four possible solutions in an infinite-order singular control case.

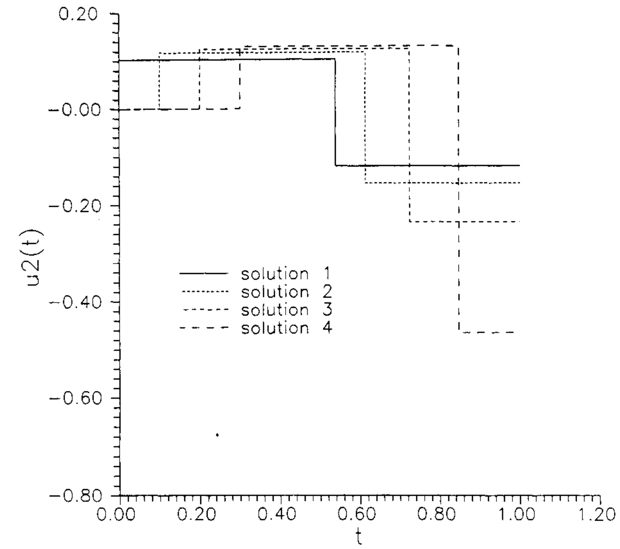

Fig. 4 Time histories of controls u_2 for four possible solutions in an infinite-order singular control case.

Note that controls u_2, u_3 are assumed nonsingular and, hence, are identically constant, except possibly at a finite number of points. Further differentiation of Eq. (37) yields

$$u_1(\omega_2d_2+\omega_3d_3)+2\omega_1(u_2d_2+u_3d_3)=0 \tag{38}$$

As u_2, u_3 are assumed nonsingular, it is clear that $\omega_2=\omega_3=0$ can occur at most at an isolated point. Hence, the coefficient of u_1 in Eq. (38) must be nonzero except possibly at an isolated point, as otherwise the second and third equation in Eq. (32) contradict the assumption $d_1^2+d_2^2+d_3^2\neq 0$. This implies that Eq. (38) can be solved for control u_1, except possibly at an isolated point, and we obtain the result stated in Eqs. (32) and (33). Kelley's necessary condition for optimality of singular control (see Refs. 12–15),

$$(-1)^q\frac{\partial}{\partial u_1}\left[\frac{d^{2q}}{dt^{2q}}\left(\frac{\partial H}{\partial u_1}\right)\right]\geq 0, \qquad q=2$$

with $q\in N$ denoting the order of the singular control, yields $\omega_2d_2+\omega_3d_3\geq 0$. The same result can also be obtained from Goh's necessary condition (see Refs. 16 and 17).

The assumption that $d_1^2+d_2^2+d_3^2=0$ immediately leads to the result stated in Eq. (31). In this case, all derivatives of S_1 are automatically zero, without ever providing a condition on control u_1 (singular control of infinite order).

B. Two Controls Singular

Without loss of generality we can assume that controls u_1 and u_2 are singular and control u_3 is nonsingular. Again, we have to distinguish two possible cases, namely, $d_1^2+d_2^2+d_3^2=0$ and $d_1^2+d_2^2+d_3^2\neq 0$.

If $d_1^2+d_2^2+d_3^2=0$, then we obtain singular control of infinite order for controls u_1 and u_2. Explicitly, we get

$$\lambda_{\omega_1}=0$$
$$\lambda_{\omega_2}=0$$
$$d_1=0$$
$$d_2=0$$
$$d_3=0 \tag{39}$$

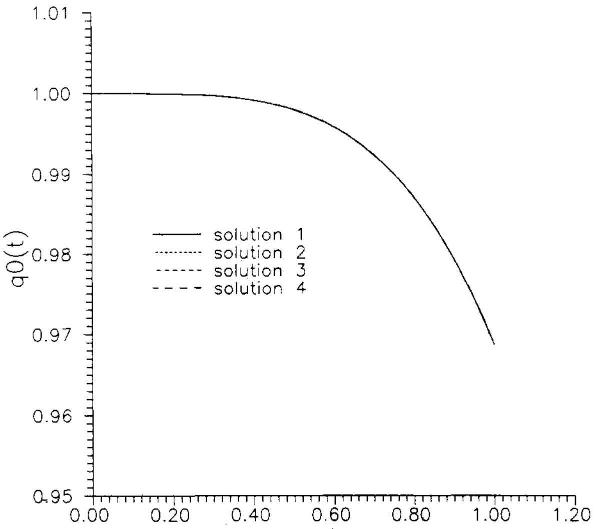

Fig. 5 Time histories of state q_0 for four possible solutions in an infinite-order singular control case.

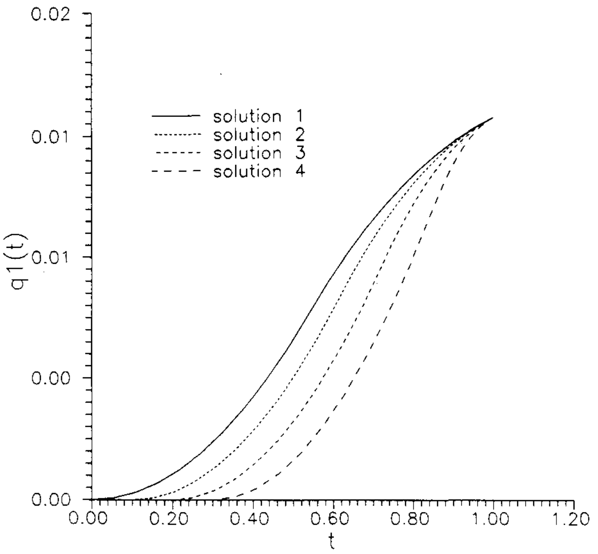

Fig. 6 Time histories of state q_1 for four possible solutions in an infinite-order singular control case.

In complete analogy to the situation discussed in the previous section, these conditions are satisfied identically throughout the trajectory, if they are only satisfied at a single point. Both controls u_1 and u_2 can be chosen arbitrarily within their allowed bounds of Eq. (2) as long as the boundary conditions of Eq. (11) are satisfied.

The case $d_1^2+d_2^2+d_3^2\neq 0$ can be excluded. Formally, conditions for a finite-order singular control can be obtained in this case, but from Goh's necessary condition it follows that this control logic is always suboptimal.

In the remainder of this section, we will give a short proof of the claims made previously. We have

$$\left.\begin{matrix} S_1=0 \\ S_2=0 \end{matrix}\right\} \overset{(27)}{\Rightarrow} \begin{matrix} \lambda_{\omega_1}=0 \\ \lambda_{\omega_1}=0 \end{matrix} \tag{40}$$

$$\left.\begin{matrix} \dot S_1=0 \\ \dot S_2=0 \end{matrix}\right\} \overset{(28)}{\Rightarrow} \begin{matrix} d_1=0 \\ d_2=0 \end{matrix} \tag{41}$$

$$\left.\begin{matrix} \ddot S_1=0 \\ \ddot S_2=0 \end{matrix}\right\} \overset{(29),(41)}{\Rightarrow} \begin{matrix} \omega_2 d_3=0 \\ \omega_1 d_3=0 \end{matrix} \tag{42}$$

If $d_1^2+d_2^2+d_3^2\neq 0$, then Eq. (42) implies

$$\omega_2=0$$
$$\omega_1=0 \tag{43}$$

and further differentiation yields explicit expressions for controls u_1, u_2, namely:

$$\left.\begin{matrix} S_1^{(3)}=0 \\ S_2^{(3)}=0 \end{matrix}\right\} \overset{(1)}{\Rightarrow} \begin{matrix} u_2=0 \\ u_1=0 \end{matrix} \tag{44}$$

It is observed that both controls u_1 and u_2 appear explicitly for the first time in $[S_1^{(3)}, S_2^{(3)}]$. In Bell and Jacobson[18] and in Kelley et al.[13] it is shown that, in the case of a single scalar control being singular, the associated switching function has to be differentiated an even number of times before the control can appear explicitly for the first time. It is also stated in Ref. 18 that, for a vector control case, "the controls can appear in an odd time derivative of $\partial H/\partial u_i$ but if this does occur then there necessarily exists a control u such that the second variation is negative (for a minimization problem)." Hence, this control logic is not optimal.

C. All Three Controls Singular

This case can be excluded for problem MTSR. We have

$$S=0 \overset{(27)}{\Rightarrow} \lambda_\omega=0 \tag{45}$$

$$\dot S=0 \overset{(28)}{\Rightarrow} d=0 \tag{46}$$

It follows immediately that all higher order derivatives of S are automatically zero. Hence, all three controls are singular of infinite order. From Eqs. (45) and (46) we find immediately that in this case the Hamiltonian, as defined in Eq. (14), satisfies $H=0$. But for a minimum time problem this contradicts the transversality condition of Eq. (21), $H|_{t_f}=-1$.

VII. Principal Axis Rotations

A rotation is called a principal axis rotation if, at every instant of time, the rotation is performed about a fixed principal axis of the spacecraft. By inspection it is observed that the results obtained in Sec. VI exclude the possibility of principal axis rotations as minimum time reorientation strategies, except

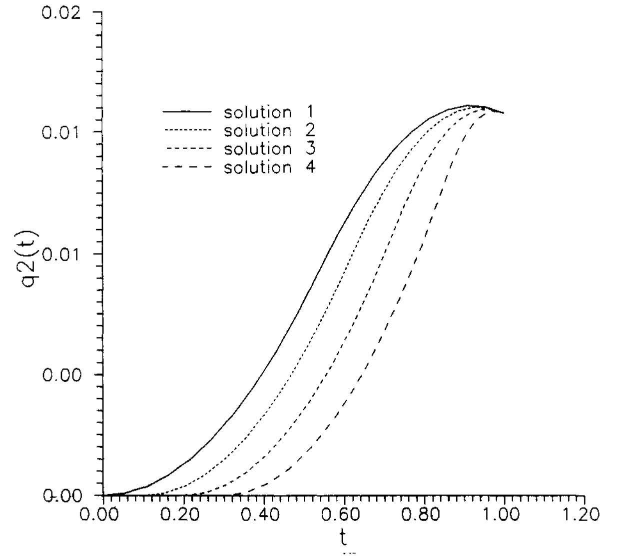

Fig. 7 Time histories of state q_2 for four possible solutions in an infinite-order singular control case.

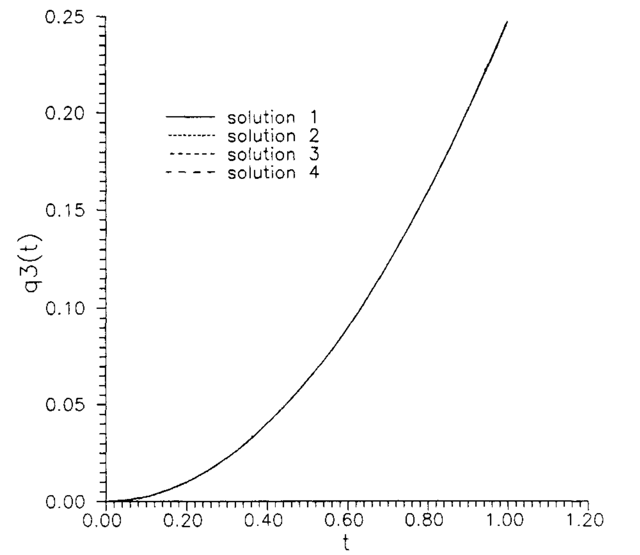

Fig. 8 Time histories of state q_3 for four possible solutions in an infinite-order singular control case.

possibly as realizations of the infinite-order singular control case of Eq. (39). For example, one such realization is given by

$$u_1(t) \equiv 0 \forall t$$

$$u_2(t) \equiv 0 \forall t$$

$$u_3(t) \equiv 1 \forall t$$

Obviously, these controls represent a principal axis rotation.

In general, it can be stated that a prinicpal axis rotation can furnish the minimum time solution only if the minimum time that it takes to satisfy all boundary conditions is equal to the minimum time that it takes to perform only the prescribed change in angular velocity. This follows immediately from the nature of the infinite-order singular control case of Eq. (39). Specifically, this implies that a principal axis rotation can never furnish the solution to a minimum time rest-to-rest reorientation problem. This contradicts a result stated earlier by Li and Bainum.[5]

VIII. Numerical Examples

In this section, numerical examples are presented, involving all of the control logics analyzed in the previous sections. The control bounds $u_{i,\max}$, $i = 1, 2, 3$ used in Eq. (2) are set equal to 1. This choice is purely for convenience.

A. Nonsingular Control

The problem of a minimum time 180-deg turn has been treated successfully in Ref. 1, and the resulting numerical solution has a switching structure of bang-bang type. The present study confirms these results.

Explicitly, the problem under consideration is given by Eqs. (1), (2), (3), and (9), subject to the initial conditions $\omega(0) = [0, 0, 0]$, $q(0) = [1, 0, 0, 0]$, the final condition $\omega(t_f) = [0, 0, 0]$, and a final condition associated with the prescribed final angular position. Even though the angular position is completely prescribed at final time (180-deg rotation with regard to initial position), not all Euler parameters can be prescribed at t_f. This is due to the lack of controllability in the dynamics of q discussed earlier in Sec. III. In the case of a 180-deg reorientation about the n_1 axis (see Fig. 1), the desired final value of

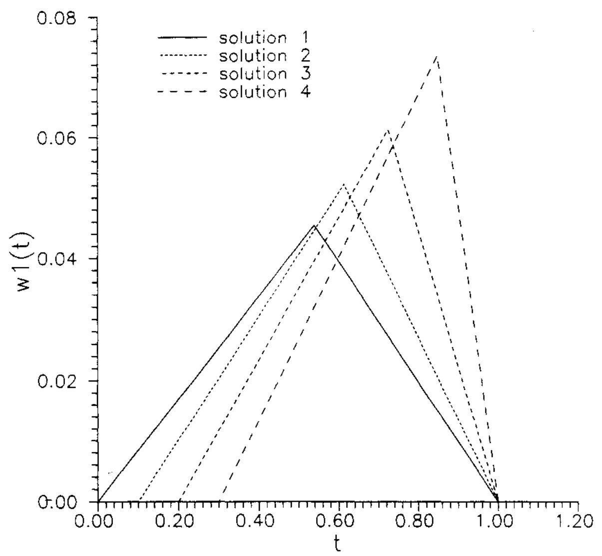

Fig. 9 Time histories of state ω_1 for four possible solutions in an infinite-order singular control case.

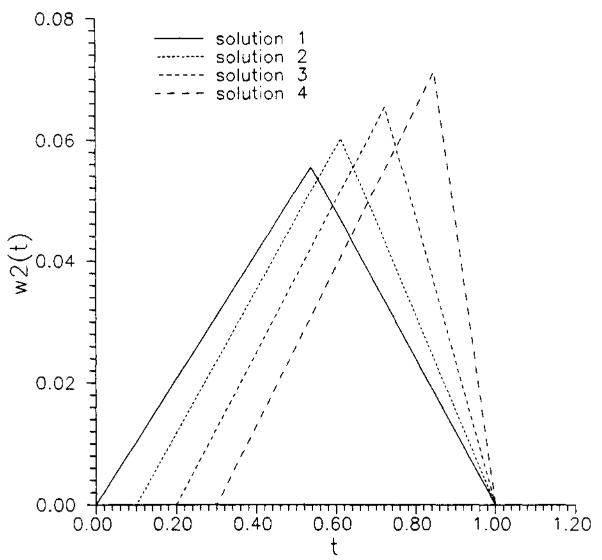

Fig. 10 Time histories of state ω_2 for four possible solutions in an infinite-order singular control case.

state q is $q(t_f) = [0, 1, 0, 0]$. To obtain a consistent boundary value problem, only $q_0(t_f) = q_2(t_f) = q_3(t_f) = 0$ can be prescribed, and $q_1(t_f)$ formally has to be considered free, which leads to the transversality condition $\lambda_{q_1}(t_f) = 0$.

If all four Euler parameters are prescribed explicitly at final time, then, depending on the software used for solving boundary value problems, it may be impossible to obtain a solution, or, at least, the convergence behavior would be very bad. The initial costate vector associated with the solution to a rest-to-rest 18-deg turn is given by

$$\lambda_{\omega_1}(0) = -0.230330$$

$$\lambda_{\omega_2}(0) = 0.076005$$

$$\lambda_{\omega_3}(0) = -0.693665$$

$$\lambda_{q_0}(0) = 0$$

$$\lambda_{q_1}(0) = -0.356038$$

$$\lambda_{q_2}(0) = 0.356038$$

$$\lambda_{q_3}(0) = -0.817612$$

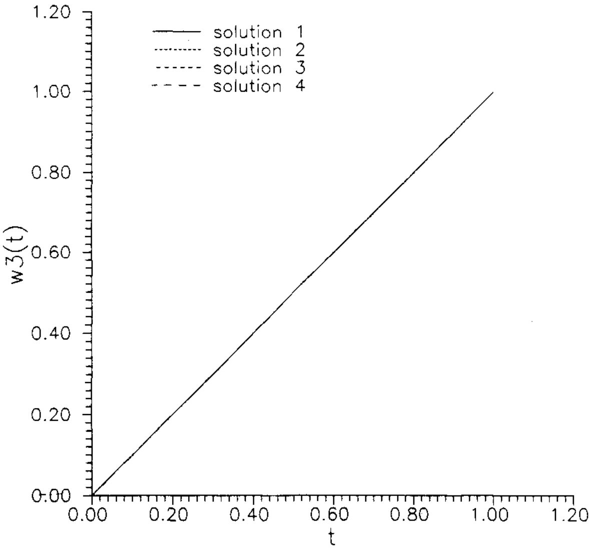

Fig. 11 Time histories of state ω_3 for four possible solutions in an infinite-order singular control case.

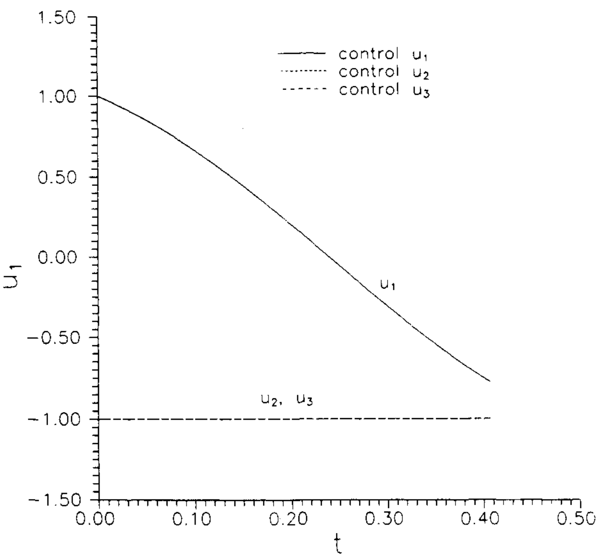

Fig. 12 Time history of controls u in a second-order singular control case.

B. Infinite-Order Singular Control

Consider the problem of Eqs. (1), (2), (3), and (9) with initial conditions

$$\omega(0)^T = [0, 0, 0]$$

$$q(0)^T = [1, 0, 0, 0]$$

and the final condition

$$\omega_3(t_f) = 1$$

Here, the costate Eq. (16), along with the transversality conditions $\lambda_q(t_f) = 0$, imply $\lambda_q(t) \equiv 0 \forall t$. Hence, with the definition of Eq. (23), we get $d \equiv 0$ on $[t_0, t_f]$, which implies that singular control of infinite order is the only possible solution to the problem (see the analysis in Sec. VI). The transversality conditions $\lambda_{\omega_1}(t_f) = 0$, $\lambda_{\omega_2}(t_f) = 0$ in conjunction with Eqs. (15) and (23) and $\dot{\lambda}_\omega = d = 0$ imply $\lambda_{\omega_1}(t) \equiv 0 \forall t$, $\lambda_{\omega_2}(t) \equiv 0 \forall t$. Finally, from the condition of Eq. (21), $H|_{t_f} = -1$, together with the switching condition of Eq. (17) and final condition $\omega_3(t_f) = 1$, we get $\lambda_{\omega_3}(t_f) = -1$. With $\dot{\lambda}_{\omega_3} = 0$, this implies $\lambda_{\omega_3}(t) = -1 \forall t$, and all Lagrange multipliers are completely determined.

Note that only control u_3 is determined explicitly by the minimum principle. Controls u_1, u_2 are arbitrary as long as the boundary conditions are satisfied. Explicitly, we get

$$u_1(t) \text{ arbitrary}$$

$$u_2(t) \text{ arbitrary}$$

$$u_3(t) \equiv 1 \forall t$$

Loosely speaking, this case of infinite-order singular control is a consequence of the decoupled structure of the controls u_i and states ω_j for $i \neq j$. Note that controls u_1, u_2 do not have any effect on angular velocity ω_3. The minimum maneuver time is solely determined by the initial/boundary conditions $\omega_3(0) = 0$, $\omega_3(t_f) = 1$, and, in return, only control u_3 is determined by the minimum principle.

Obviously, this situation is preserved if any of the final states $\omega_1(t_f)$, $\omega_2(t_f)$ are prescribed such that they can be reached within the minimum time that it takes control u_3 to drive ω_3 from 0 to 1, i.e., as long as $|\omega_3(t_f)| \geq |\omega_i(t_f)|$ for $i = 1, 2$. Then the optimal control histories are given by

$$u_1(t) \text{ arbitrary with } \int_0^1 u_1(t)\,dt = \omega_1(t_f)$$

$$u_2(t) \text{ arbitrary with } \int_0^1 u_2(t)\,dt = \omega_2(t_f)$$

$$u_3(t) \equiv 1 \forall t$$

In case of equality, i.e., $\omega_1(t_f) = \omega_3(t_f)$, or $\omega_2(t_f) = \omega_3(t_f)$, the problem is also solved by the infinite-order singular control logic discussed in Sec. VI.A, where two controls are bang-bang and one control is singular.

Another, less trivial example is obtained by prescribing the final conditions

$$q_0(t_f) \text{ free}$$

$$q_1(t_f) = 0.0126$$

$$q_2(t_f) = 0.0126$$

$$q_3(t_f) = 0.247383$$

$$\omega(t_f)^T = [0, 0, 1]$$

Physically, these boundary conditions refer to a reorientation of the satellite that is equivalent to a single 28.721-deg rotation of the satellite about an axis whose inclination with respect to the inertial axes n_1, n_2, n_3 is given by $\epsilon_1 = 87.088$ deg, $\epsilon_2 = 87.088$ deg, $\epsilon_3 = 4.120$ deg, respectively (see Figs. 1 and 2). At the same time, the satellite must be brought to an angular rate whose components in body coordinates are $\omega(t_f)^T = [0, 0, 1]$, i.e., at final time the satellite is prescribed to spin about its body z axis at an angular rate $\omega_3(t_f) = 1$.

For this problem the minimum maneuver time is again determined solely by the time that it takes control u_3 to drive angular velocity ω_3 from 0 to 1, which is 1 s. Within this time it is possible to satisfy the remaining boundary conditions in infinitely many ways, hence yielding infinite-order singular control.

Table 1 Parameters associated with four possible solutions in an infinite-order singular control case

	case 1	case 2	case 3	case 4
t_1	0.0	0.1	0.2	0.3
t_2	0.538	0.612	0.724	0.847
c_1	0.085	0.102	0.117	0.135
c_2	0.103	0.117	0.125	0.130
c_3	−0.098	−0.135	−0.222	−0.482
c_4	−0.120	−0.155	−0.236	−0.465

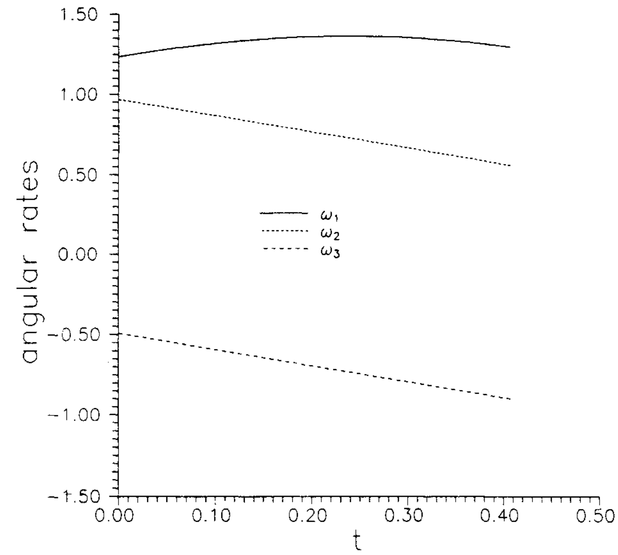

Fig. 13 Time history of angular velocities ω in a second-order singular control case.

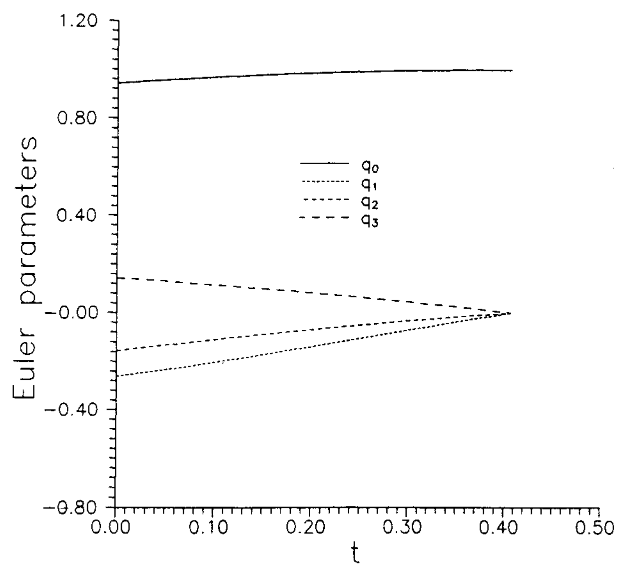

Fig. 14 Time history of Euler parameters q in a second-order singular control case.

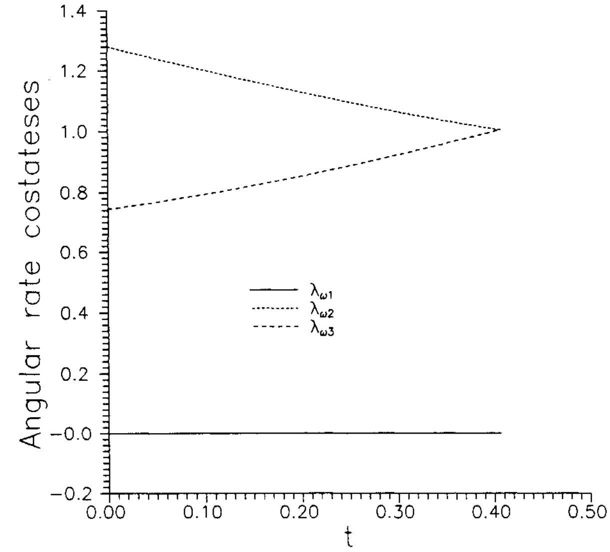

Fig. 15 Time history of costates λ_ω in a second-order singular control case.

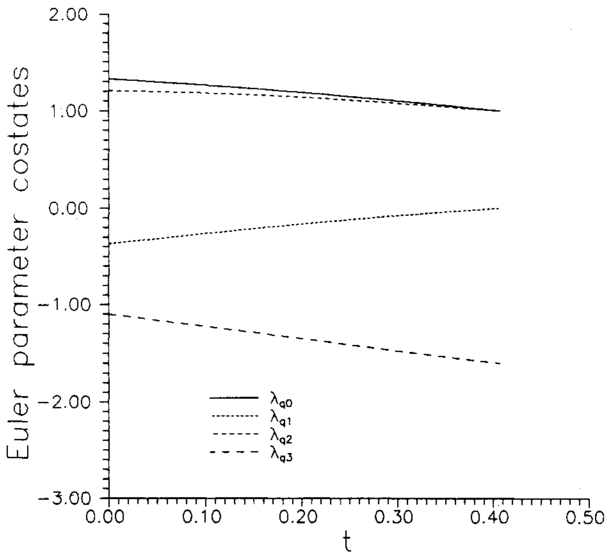

Fig. 16 Time history of costates λ_q in a second-order singular control case.

trol. A family of possible controls $u_1(t)$, $u_2(t)$ that satisfy the boundary conditions can be obtained by selecting

$$u_1(t) \equiv 0 \quad \text{on} \quad [0, t_1]$$

$$u_1(t) \equiv c_1 \quad \text{on} \quad [t_1, t_2]$$

$$u_1(t) \equiv c_2 \quad \text{on} \quad [t_2, 1]$$

$$u_2(t) \equiv 0 \quad \text{on} \quad [0, t_1]$$

$$u_2(t) \equiv c_3 \quad \text{on} \quad [t_1, t_2]$$

$$u_2(t) \equiv c_4 \quad \text{on} \quad [t_2, 1]$$

Control u_3 is identically 1.

In Table 1 the numerical values for the parameters t_1, t_2, c_1, c_2, c_3, c_4 are given for four selected cases, namely, $t_1 = 0.0$, $t_1 = 0.1$, $t_1 = 0.2$, and $t_1 = 0.3$. In Figs. 3–11 the associated time histories for controls u_1, u_2, states ω, q, and costates λ_ω, λ_q are displayed.

C. Finite-Order Singular Control

The only possible control logic involving finite-order singular control is given in Sec. VI.A. Possibly after a permutation of the axis, it is of the form: u_1 singular of second order and u_2, u_3 bang-bang. At the beginning of a singular arc of this type, conditions of Eq. (32) have to be satisfied, and, in the interior of the singular arc, control u_1 is given by Eq. (33).

For a numerical example, consider the problem of Eqs. (1), (2), (3), and (9) with initial conditions

$$\omega(0)^T = [-0.045,\ -1.1,\ -1.0]$$

$$q(0)^T = [1, 0, 0, 0]$$

and final conditions

$$\omega(t_f)^T = [-0.472,\ -1.1,\ -1.899]$$

$$q(t_f)^T = [-0.274, 0.076,\ -0.356, \text{free}]$$

These boundary conditions are chosen such that the resulting trajectory is singular throughout the time interval. The optimal controls, as well as state and costate functions of time, are given in Figs. 12–16.

IX. Conclusions

The minimum time reorientation problem for an inertially symmetric rigid spacecraft with three bounded independent control torques aligned with the principal axes is investigated. All possible control logics are clearly identified. These include bang-bang type control and singular control of finite and infinite order. Numerical examples are provided for all cases.

Principal axis rotations for rest-to-rest maneuvers are proved to be suboptimal.

Furthermore, the concept of singular control of infinite order is addressed in this paper, and it is shown that for certain problems all controls that furnish minimum cost are of this type.

Appendix

In the following we present a simple optimal control problem for which, in general, singular control of infinite order furnishes the only solution. Consider the problem

$$\min_{u,v \in PWC[t_0,t_f]} t_f \tag{A1}$$

subject to the dynamical system

$$\dot{x} = u$$
$$\dot{y} = v \tag{A2}$$

the boundary conditions

$$x(0) = 0, \qquad x(t_f) = 0$$
$$y(0) = 0, \qquad y(t_f) = 1 \tag{A3}$$

and the control constraints

$$u \in [-1,\ +1]$$
$$v \in [-1,\ +1] \tag{A4}$$

It is easy to see that the dynamics of states x and y are completely decoupled and the minimum final time for the problem of Eqs. (A1–A4) is exactly the minimum time that it takes to drive state y to its prescribed final value $y(t_f) = 1$. For control u we can choose $u(t) \equiv 0$, but any other choice is also possible as long as $x(t_f) = 0$ remains satisfied. Hence, we see that there is a well-defined minimum cost associated with the problem of Eqs. (A1–A4). Even though the controls that furnish this minimum cost are not determined uniquely, it may very well be of engineering interest to determine one set of control functions that furnishes this minimum cost.

It is also clear that any solution to Eqs. (A3) and (A4) that minimizes the cost criterion has to satisfy the first-order necessary conditions of optimal control. Applying these conditions, we find the costate equations

$$\dot{\lambda}_x = 0$$
$$\dot{\lambda}_y = 0 \tag{A5}$$

that means both costates are constant throughout the time interval $[t_0, t_f]$. In the nonsingular case $\lambda_x \neq 0$, $\lambda_y \neq 0$, the controls u, v are determined by

$$u = \begin{cases} +1 & \text{if} \quad \lambda_x < 0 \\ -1 & \text{if} \quad \lambda_x > 0 \end{cases} \tag{A6}$$

$$v = \begin{cases} +1 & \text{if} \quad \lambda_y < 0 \\ -1 & \text{if} \quad \lambda_y > 0 \end{cases} \tag{A7}$$

It is clear that the problem of Eqs. (A1–A4) cannot be solved without using singular control, as the boundary value problem of Eqs. (A2), (A5), and (A3) with the control logic of Eqs. (A6) and (A7) does not have a solution for $\lambda_x \neq 0$, $\lambda_y \neq 0$.

Let us now consider the case where control u is singular. Control u is singular if and only if the associated switching function $S_u := \lambda_x$ is identically zero. From Eqs. (A5) it is clear that all time derivatives of S_u are automatically zero. In particular, this implies that the condition $S_u \equiv 0$ can never provide any condition on control u. If S_u is zero at a single point along the trajectory, then S_u will be identically zero throughout the trajectory. The stationary nature of the cost criterion is not influenced by the choice of control u, in this case. Any control function of time $u(t)$ can be chosen as long as it steers the state to the prescribed final position.

References

[1]Bilimoria, K. D., and Wie, B., "Minimum-Time Large-Angle Reorientation of a Rigid Spacecraft," *Proceedings of the AIAA Guidance, Navigation, and Control Conference* (Portland, OR), AIAA, Washington, DC, Aug. 1990 (AIAA Paper 90-3486).

[2]Chowdhry, R. S., and Cliff, E. M., "Optimal Rigid Body Reorientation Problem," *Proceedings of the AIAA Guidance, Navigation, and Control Conference* (Portland, OR), AIAA, Washington, DC, Aug. 1990 (AIAA Paper 90-3485).

[3]Junkins, J. L., and Turner, J. P., *Optimal Spacecraft Rotational Maneuvers,* Elsevier, New York, 1985.

[4]Junkins, J. L., and Turner, J. P., "Optimal Continuous Torque Attitude Maneuvers," *Journal of Guidance and Control,* Vol. 3, No. 3, 1980, pp. 210–217.

[5]Li, F., and Bainum, P. M., "Numerical Approach for Solving Rigid Spacecraft Minimum-Time Attitude Maneuvers," *Journal of Guidance, Control, and Dynamics,* Vol. 13, No. 1, 1990, pp. 38–45.

[6]Hermes, H., and Lasalle, J. P., *Functional Analysis and Time Optimal Control,* Academic, New York, 1969.

[7]Haynes, G. W., "Controllability of Nonlinear Systems with Linearly Occurring Controls," NASA CR-1463, NASA Ames Research Center, 1969.

[8]Bryson, A. E., and Ho, Y. C., *Applied Optimal Control,* Hemisphere, New York, 1975.

[9]Ewing, G. M., *Calculus of Variations,* Dover, New York, 1985.

[10]Lee, E. B., and Markus, L., *Foundations of Optimal Control Theory,* Krieger, Malabar, FL, 1986.

[11]Neustadt, L. W., *A Theory of Necessary Conditions,* Princeton Univ. Press, Princeton, NJ, 1976.

[12]Kelley, H. J., "A Second Variation Test for Singular Extremals," *AIAA Journal,* Vol. 2, No. 8, 1964, pp. 1380–1382.

[13]Kelley, H. J., Kopp, R. E., and Moyer, H. G., "Singular Extremals in Optimal Control," *Optimization Techniques,* edited by G. Leitmann, Vol. 2, Academic, New York, 1967.

[14]Kopp, R. E., and Moyer, H. G., "Necessary Conditions for Singular Extremals," *AIAA Journal,* Vol. 3, No. 8, 1965, pp. 1439–1444.

[15]Robbins, H. M., "A Generalized Legendre-Clebsch Condition for the Singular Cases of Optimal Control," *IBM Journal,* Vol. 11, No. 4, July 1967, pp. 361–372.

[16]Goh, B. S., "Necessary Conditions for Singular Extremals Involving Multiple Control Variables," *SIAM Journal of Control,* Vol. 4, No. 4, 1966, pp. 716–731.

[17]Seywald, H., "Optimal Control Problems with Switching Points," Ph.D. Dissertation, Virginia Polytechnic Inst. and State Univ., Blacksburg, VA, July 1990.

[18]Bell, D. J., and Jacobson, D. H., "Singular Optimal Control Problems," *Mathematics in Science and Engineering,* Academic, New York, 1975.